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Note

Minimum cutsets in hypercubes

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Abstract

A *local cut* at a vertex v is a set consisting of, for each neighbor x of v, the vertex x or the edge vx. We prove that the local cuts are the smallest sets of vertices and/or edges whose deletion disconnects the k-dimensional hypercube Q_k. We also characterize the smallest sets of vertices and/or edges whose deletion produces a graph with larger diameter than Q_k. These are the sets consisting of $k-1$ elements from a local cut.

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1. Introduction

It is an elementary exercise that the connectivity and the edge-connectivity of the k-dimensional hypercube Q_k both equal k (see [8] for definitions). The neighborhood $N(v)$ of any vertex v is a minimum vertex cut in Q_k, and the set of edges incident to a vertex forms a minimum edge cut.

Define a *generalized cut* of a graph G to be a set U of vertices and/or edges such that $G-U$ is disconnected. In Section 2, we characterize the smallest generalized cuts of Q_k using a general lemma about the relationship between such sets and minimum vertex cuts. The general lemma states that for simple graphs with equal connectivity and edge-connectivity

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(other than complete graphs), every smallest generalized cut has size $\kappa(G)$ and is obtained from a minimum separating set by possibly replacing some of the vertices with incident edges.

Let $N(v)$ denote the set of neighbors of a vertex v in a graph G; its size is the vertex degree $d(v)$. For a vertex v in G, a *local cut* at v is a set of size $d(v)$ consisting of the vertex x or the edge vx for each $x \in N(v)$. When the connectivity $\kappa(G)$ and the edge-connectivity $\kappa'(G)$ both equal the minimum vertex degree $\delta(G)$, every local cut at a vertex of minimum degree is a smallest generalized cut. If also G is triangle-free, and $\delta(G) \geqslant 3$, and if (as in Q_k), closed vertex neighborhoods are not separating sets, then every smallest generalized cut has this form. (The *closed neighborhood* $N[v]$ of a vertex v in G is $N(v) \cup \{v\}$.)

A disconnected graph has infinite diameter. Thus, Section 2 studies the smallest sets whose deletion sends the diameter of the remaining graph to ∞. In Section 3, we study the smallest sets whose deletion from Q_k yields a remaining graph with larger diameter; that is, larger than k. These results are limited to Q_k. We use $d(x, y)$ for the distance between x and y in a graph G, and the *diameter* is $\max\{d(x, y) : x, y \in V(G)\}$.

There has been some study of related questions. The n-cycle C_n shows that it is possible to remove a single vertex from a 2-connected graph and nearly double the diameter. More generally, Chung and Garey [2] proved that the maximum diameter achievable by removing $k - 1$ edges from a k-edge-connected graph with diameter D is $kD + O(k)$.

The hypercube is far from this extreme; in some sense, Q_k is a very highly interconnected k-connected graph. We show that deleting $k - 2$ vertices and/or edges cannot increase the diameter, deleting $k - 1$ can increase it by at most 1, and the sets of size $k - 1$ that increase it by 1 are the sets obtained from local cuts by deleting one element. Our analysis is based on the existence of many pairwise internally disjoint short paths joining arbitrary pairs of vertices.

Graphs in which at least k vertices must be deleted to increase the distance between at least one pair of remaining vertices are studied in [3].

2. Generalized cuts

We begin by characterizing the minimum separating sets (of vertices) in Q_k. This fact is presumably folklore; it uses a slightly more careful look at a standard inductive proof that $\kappa(Q_k) = k$ (see Example 4.1.3 of [8]).

Proposition 1. *For $k \geqslant 2$, the only separating sets of size at most k in Q_k are vertex neighborhoods.*

Proof. This holds by inspection for $k = 2$. For $k > 2$, let Q' and Q'' be two copies of Q_{k-1} joined by a matching to form Q_k. Let S be a minimum separating set of Q_k. If $Q' - S$ and $Q'' - S$ are both connected, then S must include a vertex from each edge of the matching between Q' and Q''; this yields $|S| \geqslant 2^{k-1} > k$.

Hence, we may assume that $Q' - S$ is disconnected and apply the induction hypothesis. If $Q'' - S$ is also disconnected, then $|S| \geqslant 2k - 2 > k$. If $S \cap V(Q'') = \emptyset$, then $Q_k - S$ is connected. Hence, we may assume that $|S \cap V(Q')| \geqslant k - 1$ and $|S \cap V(Q'')| \geqslant 1$. Furthermore, equality requires $S \cap V(Q')$ to be a vertex neighborhood, say $N_{Q'}(v)$. This means that $Q' - S$ has two components. Because Q'' is 2-connected, the only way to delete one vertex to break all paths through Q'' that join v to one of its nonneighbors in Q' is to delete the neighbor of v in Q''. Hence S has the desired form. □

Now we consider deletion of both vertices and edges in arbitrary simple graphs. The next lemma uses a slightly more careful look at a standard proof of Whitney's result that $\kappa(G) \leqslant \kappa'(G)$ for every graph G (see Theorem 4.1.9 of [8]). We exclude complete graphs which have no cuts.

Lemma 2. *If G is a simple graph with $\kappa(G) = \kappa'(G) < n(G) - 1$, then every smallest generalized cut in G has size $\kappa(G)$ and consists of a subset of a minimum separating set and one edge incident to each remaining vertex of that separating set.*

Proof. Let U be a smallest generalized cut of G, and let $A = U \cap V(G)$ and $B = U \cap E(G)$. Let $G' = G - A$. By the choice of U, B is a minimum edge cut of G'. We write this as $B = [S, S']$ for some partition S, S' of $V(G')$.

There must be some vertices $x \in S$ and $y \in S'$ such that x and y are not adjacent, since otherwise $|B| = |S|(n(G') - |S|) \geqslant n(G') - 1$. Let T be the vertex set consisting of all neighbors of x in S' and all vertices of $S - \{x\}$ having neighbors in S'.

Note that T is an x, y-separating set in G'. Furthermore, $|[S, S']| \geqslant |T|$, with equality if and only if the cut consists only of the edges from x to S' (one to each vertex of $T \cap S'$) and one edge from each vertex of $T \cap S$ to S' (these may lead to $T \cap S'$).

Since $A \cup T$ is a vertex cut in G, and $|B| \geqslant |T|$, we have $|U| \geqslant \kappa(G)$. Hence equality holds, and U is as described. □

Proposition 3. *Let G be a triangle-free graph with $3 \leqslant \kappa(G) = \kappa'(G) = \delta(G) < n(G) - 1$. If all minimum vertex cuts are vertex neighborhoods and the subgraphs obtained by deleting closed neighborhoods of vertices of minimum degree are connected, then every smallest generalized cut of G is a local cut.*

Proof. Let U be a smallest generalized cut in G, and let A be the set of vertices and B the set of edges in U. By Lemma 2 and the hypothesis of this proposition that all minimum vertex cuts are neighborhoods of vertices of minimum degree, A is a subset of the neighborhood of a single vertex v, and B consists of one edge incident to each remaining neighbor of v.

By hypothesis, $G - N[v]$ is connected. Since G is triangle-free and $\kappa(G) \geqslant 3$, each vertex of $N(v)$ retains a neighbor outside $N[v]$ in $G - U$. If B lacks the edge from v to some $u \in N(v) - A$ and instead has an edge from u to a vertex outside $N[v]$, then in $G - U$ there is an edge from v to u, and $G - U$ is connected, contradicting the assumption that U is a generalized cut in G.

We conclude that U is a local cut. □

Corollary 4. *For $k \geqslant 3$, every smallest generalized cut in Q_k is a local cut.*

Proof. For $k \geqslant 3$, Q_k satisfies all the hypotheses of Proposition 3. □

When $k = 2$, a perfect matching is a minimum edge cut and minimum generalized cut that is not a local cut.

3. Diameter invulnerability

We restrict our study of diameter under deletion of vertices and edges to the hypercube Q_k. Still, we use one general remark.

Remark. If x and y are vertices in a graph G, and G has k pairwise internally disjoint x, y-paths of length at most s, then at least k vertices and/or edges must be deleted to increase the distance between x and y above s.

Letting x and y be vertices in Q_k with $d(x, y) = r$, we apply the Remark with $s = r + 2$ when $r < k$ and with $s = r$ when $r = k$.

Lemma 5. *Suppose that $k \geqslant 3$. Given vertices x and y in Q_k with $d(x, y) = r$, there exist k pairwise internally disjoint x, y-paths of which r have length r and $k - r$ have length $r + 2$.*

Proof. We refer to vertices by their k-tuple names. By symmetry, we may assume that x is the 0-vector and that y differs from it in the first r coordinates. Paths are described by successive changes in single coordinates. We obtain a canonical x, y-path of length r via successively changing, in order, coordinates $1, 2, \ldots, r$. The $r - 1$ other cyclic permutations of this list of coordinates produce $r - 1$ other x, y-paths of length r. The internal vertices on the ith path have 1 in position i and 0 in position $i - 1$ (modulo r). Hence the paths are pairwise internally disjoint, since the 1s in each internal vertex occur consecutively (modulo r).

For $r+1 \leqslant i \leqslant k$, the list of coordinates that change in the ith path, in order, is $i, 1, 2, \ldots, r, i$. The length is$r + 2$, and the internal vertices have 1 in position i and nowhere else among positions $r + 1, \ldots, k$. $\square$

By “objects”, we mean “vertices or edges”.

Lemma 6. *If $k \geqslant 3$, and U is a smallest set of vertices and edges in Q_k whose deletion destroys all shortest paths joining two antipodal vertices, then U is a local cut at one of those vertices.*

Proof. By symmetry, we may let x be the k-tuple of 0s and y be the k-tuple of 1s. There are $k!$ paths to destroy. Each deleted vertex of weight r lies on $r!(k - r)!$ paths. Each edge from a vertex of weight r to a vertex of weight $r + 1$ lies on $r!(k - r - 1)!$ paths. The total number of paths destroyed by k objects is at most $k(k - 1)!$, with equality if and only if no path is destroyed twice and all the objects belong to the local cuts at x and y.

Furthermore, an object in the local cut at x and an object in the local cut at y lie on a common path unless the vertices in $N(x)$ and $N(y)$ that involve them are complements. Suppose that the cut is not wholly contained in either the local cut at x or the local cut at y. Then since $k \geqslant 3$, at least two of the objects belong to the same local cut, and at least one belongs to the other. Say two of them are in the local cut at x. Thus, the two vertices in $N(x)$ must both be the complement of the vertex in $N(y)$, contradicting the fact that each vertex has a unique complement. □

Let a *diameter-increasing set* in a graph G be a set U of vertices and edges such that the diameter of $G - U$ exceeds the diameter of G.

Theorem 7. *If* $k \geqslant 4$, *and* U *is a smallest diameter-increasing set in* Q_k, *then the diameter of* $Q_k - U$ *is* $k+1$, *and* U *consists of* $k-1$ *elements in a local cut of* Q_k. *Furthermore,* $\operatorname{diam}(Q_k - U)$ *does exceed* k *for each such set.*

Proof. We first remark that the deletion of any set U that consists of $k-1$ elements in a local cut does increase the diameter. If the local cut is at vertex v, deleting $k-1$ elements from it leaves only one vertex u adjacent to v. The distance from u to its complement u' is still k, and every v, u'-path in $Q_k - U$ consists of the edge vu plus a u, u'-path. Hence the distance from v to u' is $k+1$ in $Q_k - U$.

Now consider two vertices $x, y \in Q_k$. By Lemma 5 and the Remark, at least k objects must be deleted to break all x, y-paths of length at most k if $d(x, y) \leqslant k-2$. If $d(x, y) = k$, then by Lemma 5 there are k x, y-paths of length k and by the Remark at least k vertices and/or edges must be deleted to increase the distance between this pair beyond k. If $d(x, y) = k-1$, then at least $k-1$ objects must be deleted to break the paths of length $k-1$ and reach distance $k+1$.

Therefore, every smallest diameter-increasing set U in Q_k has size $k-1$ and causes two vertices x and y at distance $k-1$ in Q_k to be separated by distance $k+1$. Let Q' be a subgraph of Q_k that contains x and y and is isomorphic to Q_{k-1}. The set U breaks all x, y-paths of length $k-1$ in Q'. By Lemma 6, U consists of $k-1$ objects in Q' and is a local cut at x or y in Q'. By Lemma 5, this makes U a subset of local cut in Q_k, as claimed. □

Since Lemma 5 is not valid for Q_2, the last step of the proof of Theorem 7 is not valid when $k = 3$. Still, $|U| \geqslant 2$, but there is another type of smallest diameter-increasing set. In particular, deleting from Q_3 the edge $\{010, 110\}$ and the edge $\{000, 100\}$ (or just its endpoint 100) makes the distance from 000 to 110 increase to 4.

4. Remark

Other papers in the literature concerned with the behaviour of the diameter of a graph under edge deletion are [1], [4], [5], [6] and [7].

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